Notes for AA214, Chapter 13
ANALYSIS OF SPLIT AND FACTORED FORMS

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Representative Equation for Circulant Operators

1. Linear PDE's with constant coefficients with periodic boundary.
2. Analysis *depends critically on the fact that all circulant matrices commute and have a common set of eigenvectors.*
3. Assume a split PDE-ODE system with two or more splittings.
4. Representative Equation: A_a and A_b are circulant matrices.

$$\frac{d\vec{u}}{dt} = A_a\vec{u} + A_b\vec{u} - \vec{f}(t) \tag{1}$$

Scalar Split Representative Equation

Diagonalization arguements (taking advantage of the common eigenvector of circulant matrices):

The representative equation for split, circulant systems is

$$\frac{du}{dt} = [\lambda_a + \lambda_b + \lambda_c + \cdots]u + ae^{\mu t}$$

where $\lambda_a + \lambda_b + \lambda_c + \cdots$ is the sum of the eigenvalues in A_a , A_b , A_c , $\cdots$ that share the same eigenvector.

Example Analysis of Circulant Systems

1. Linear convection-diffusion equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{2}$$

2. Periodic 2^{nd} order central differences

$$\frac{d\vec{u}}{dt} = -\frac{a}{2\Delta x}B_p(-1,0,1)\vec{u} + \frac{\nu}{\Delta x^2}B_p(1,-2,1)\vec{u} \tag{3}$$

3. Represented by the eigenvalues λ_c and λ_d: $\theta_m = 2m\pi/M$.

$$\begin{aligned} (\lambda_c)_m &= \frac{-ia}{\Delta x}\sin\theta_m \\ (\lambda_d)_m &= -\frac{4\nu}{\Delta x^2}\sin^2\frac{\theta_m}{2} \end{aligned} \tag{4}$$

The Explicit-Implicit Method Analysis

1. Explicit-Implicit Method:

$$\begin{aligned} \tilde{u}_{n+1} &= [\, I \ + hA_c]\vec{u}_n \\ [\, I \ - hA_d]\vec{u}_{n+1} &= \tilde{u}_{n+1} \end{aligned} \tag{5}$$

2. Applied to $u^{'} = (\lambda_d + \lambda_c)u + ae^{\mu t}$

$$P(E) = (1 - h\lambda_d)E - (1 + h\lambda_c)$$

3. This leads to the principal σ root

$$\sigma = \frac{1 - i\dfrac{ah}{\Delta x}\sin\theta_m}{1 + 4\dfrac{h\nu}{\Delta x^2}\sin^2\dfrac{\theta_m}{2}}$$

4. Define the dimensionless numbers

$$\begin{aligned} C_n &= \frac{ah}{\Delta x}, & \text{Courant number} \\ R_\Delta &= \frac{a\Delta x}{\nu}, & \text{mesh Reynolds number} \end{aligned}$$

5. Absolute value of σ

$$|\sigma| = \frac{\sqrt{1 + C_n^2 \sin^2 \theta_m}}{1 + 4\dfrac{C_n}{R_\Delta} \sin^2 \dfrac{\theta_m}{2}} \quad , \quad 0 \le \theta_m \le 2\pi \tag{6}$$

6. Critical range of θ_m for any combination of C_n and R_Δ occurs when θ_m is near 0 (or 2π).

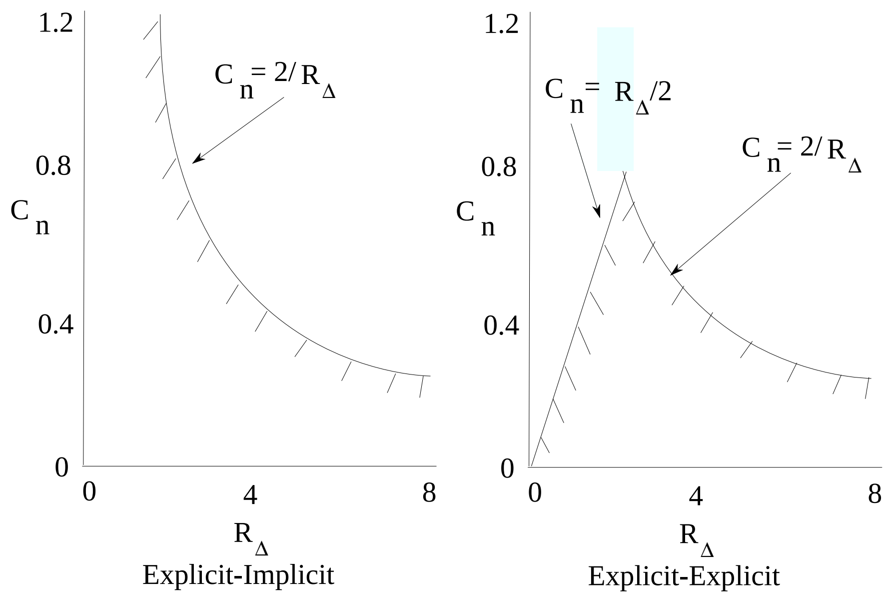
7. Condition on C_n and R_Δ that makes $|\sigma| \approx 1$ is

$$\left[1 + C_n^2 \sin^2 \epsilon\right] = \left[1 + 4\frac{C_n}{R_\Delta} \sin^2 \frac{\epsilon}{2}\right]^2$$

8. As $\epsilon \to 0$ this gives the stability region

$$C_n < \frac{2}{R_\Delta}$$

9. Bounded by a hyperbola



$$C_n = \frac{ah}{\Delta x}, R_\Delta = \frac{a\Delta x}{\nu}$$

The Explicit-Explicit Method

1. Explicit-Explicit Method:

$$
\begin{aligned}
\tilde{u}_{n+1} &= [\, I \ + hA_c]\vec{u}_n \\
\vec{u}_{n+1} &= [\, I \ + hA_d]\tilde{u}_{n+1}
\end{aligned}
\tag{7}
$$

2. An analysis similar to the one given above

$$
|\sigma| = \sqrt{1 + C_n^2 \sin^2 \theta_m}\left[1 - 4\frac{C_n}{R_\Delta}\sin^2\frac{\theta_m}{2}\right]
$$

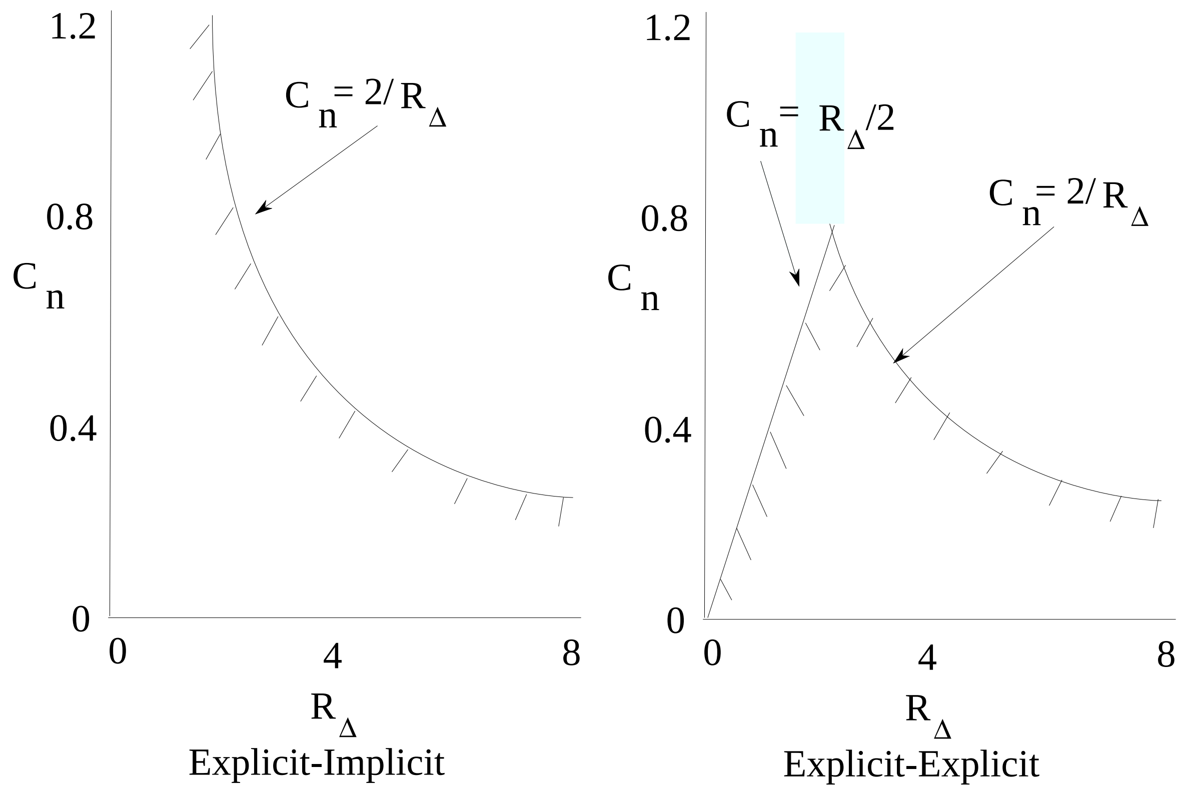
3. Two critical ranges of θ_m

   (a) Near 0: yields the same result as in the previous example

   (b) Near 180^o: produces the constraint that

$$C_n < \frac{1}{2} R_\Delta \qquad \text{for} \quad R_\Delta \leq 2$$

4. The totally explicit factored method has a much smaller region of stability when R_Δ is small, as expected.



$$C_n = \frac{ah}{\Delta x}, R_\Delta = \frac{a\Delta x}{\nu}$$

Representative Equation for Space-Split Operators

1. 2-D model equations for the space vector U.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{8}$$

$$\frac{\partial u}{\partial t} + a_x \frac{\partial u}{\partial x} + a_y \frac{\partial u}{\partial y} = 0 \tag{9}$$

2. Spatial differencing approximations: coupled set of ODE's

$$\frac{dU}{dt} = [A_x + A_y]U - f \tag{10}$$

The 2–D representative equation for model linear systems is

$$\frac{du}{dt} = [\lambda_x + \lambda_y]u + ae^{\mu t}$$

where λ_x and λ_y are *any combination* of eigenvalues from A_x and A_y, a and μ are constants.

Exact Solution to 2D Representative Equation

1. Restrict to the study of convergence rates and steady-state solutions, let $\mu = 0$

$$\frac{du}{dt} = [\lambda_x + \lambda_y]u + a \tag{11}$$

2. Exact solution

$$u(t) = ce^{(\lambda_x+\lambda_y)t} - \frac{a}{\lambda_x + \lambda_y} \tag{12}$$

The Unfactored Implicit Euler Method

1. Unfactored, first-order scheme

$$[I - hA_x - hA_y]U_{n+1} = U_n - hf \tag{13}$$

2. Applied to the representative equation

$$(1 - h\,\lambda_x - h\,\lambda_y)u_{n+1} = u_n + ha$$

3. $O\Delta E$ Analysis

$$\begin{aligned} P(E) &= (1 - h\,\lambda_x - h\,\lambda_y)E - 1 \\ Q(E) &= h \end{aligned} \tag{14}$$

4. $O\Delta E$ solution

$$u_n = c\left[\frac{1}{1 - h\,\lambda_x - h\,\lambda_y}\right]^n - \frac{a}{\lambda_x + \lambda_y}$$

5. Like its counterpart in the 1-D case, this method:
   (a) Is unconditionally stable.
   (b) Produces the exact (see Eq. 12) steady-state solution (of the ODE) for any h.
   (c) Converges very rapidly to the steady state when h is large.
6. Impractical for large multi-dimensional problems.

Factored Nondelta Form: Implicit Euler Method

1. Factored Euler method

$$[I - hA_x][I - hA_y]U_{n+1} = U_n - hf \tag{15}$$

2. Applying the representative equation.

$$(1 - h\,\lambda_x)(1 - h\,\lambda_y)u_{n+1} = u_n + ha$$

3. $O\Delta E$ Analysis

$$\begin{aligned} P(E) &= (1 - h\,\lambda_x)(1 - h\,\lambda_y)E - 1 \\ Q(E) &= h \end{aligned} \tag{16}$$

4. $O\Delta E$ solution

$$u_n = c\left[\frac{1}{(1 - h\,\lambda_x)(1 - h\,\lambda_y)}\right]^n - \frac{a}{\lambda_x + \lambda_y - h\lambda_x\lambda_y}$$

   (a) Is unconditionally stable.

   (b) Steady-state solution that depends on the choice of h.

   (c) Converges rapidly to a steady state for large h, but the converge solution is completely wrong.

5. Requires far less storage and work than the unfactored form.

6. Transient solution is only first-order accurate

Factored Delta Form of the Implicit Euler Method

1. Delta form

$$[I - hA_x][I - hA_y]\Delta U_n = h[A_{x+y}U_n + f] \tag{17}$$

2. 2-D representative equation

$$(1 - h\,\lambda_x)(1 - h\,\lambda_y)(u_{n+1} - u_n) = h(\lambda_x u_n + \lambda_y u_n + a)$$

3. $O\Delta E$ analysis

$$\begin{aligned} P(E) &= (1 - h\,\lambda_x)(1 - h\,\lambda_y)E - (1 + h^2\lambda_x\lambda_y) \\ Q(E) &= h \end{aligned} \tag{18}$$

4. $O\Delta E$ Solution

$$u_n = c\left[\frac{1 + h^2\lambda_x\lambda_y}{(1 - h\,\lambda_x)(1 - h\,\lambda_y)}\right]^n - \frac{a}{\lambda_x + \lambda_y}$$

   (a) Is unconditionally stable.
   (b) Produces the exact steady-state solution for any choice of h.
   (c) Converges very slowly to the steady–state solution for large values of h, since $|\sigma| \to 1$ as $h \to \infty$.

5. Like the factored nondelta form, this method demands far less storage and work than the unfactored form

6. Correct steady solution is obtained

7. Convergence is not as rapid as that of the unfactored form.

Analysis of the 3-D Model Equation

1. 3D representative equation

$$\frac{du}{dt} = [\lambda_x + \lambda_y + \lambda_z]u + a \tag{19}$$

2. Analyze a 2nd-order accurate trapezoidal method

$$u_{n+1} = u_n + \frac{1}{2}h[(\lambda_x + \lambda_y + \lambda_z)u_{n+1} + (\lambda_x + \lambda_y + \lambda_z)u_n + 2a]$$

3. Delta Form:

$$\left[1 - \frac{1}{2}h(\lambda_x + \lambda_y + \lambda_z)\right]\Delta u_n = h[(\lambda_x + \lambda_y + \lambda_z)u_n + a]$$

4. Factored three space directions

$$\begin{aligned}\left(1 - \frac{1}{2}h\lambda_x\right)\left(1 - \frac{1}{2}h\lambda_y\right)\left(1 - \frac{1}{2}h\lambda_z\right)\Delta u_n = \\ h\left[(\lambda_x + \lambda_y + \lambda_z)u_n + a\right]\end{aligned} \tag{20}$$

5. Preserves second order accuracy, error terms are both $O(h^3)$.

$$\frac{1}{4}h^2(\lambda_x\lambda_y + \lambda_x\lambda_z + \lambda_y\lambda_z)\Delta u_n \quad \text{and} \quad \frac{1}{8}h^3\lambda_x\lambda_y\lambda_z$$

6. $O\Delta E$ Solution

$$u_n = c\left[\frac{\left(1+\frac{1}{2}h\lambda_x\right)\left(1+\frac{1}{2}h\lambda_y\right)\left(1+\frac{1}{2}h\lambda_z\right) - \frac{1}{4}h^3\lambda_x\lambda_y\lambda_z}{\left(1-\frac{1}{2}h\lambda_x\right)\left(1-\frac{1}{2}h\lambda_y\right)\left(1-\frac{1}{2}h\lambda_z\right)}\right]^n - \frac{a}{\lambda_x+\lambda_y+\lambda_z} \tag{21}$$

7. 3D Factored Delta-Form Implicit Method
   (a) Method converges to exact steady-state
   (b) If the λ's
      i. Real negative (Diffusion): Unconditionally *Stable*
      ii. Pure imaginary (Convection): Unconditionally *UnStable*
      iii. Complex (Convection-Diffusion-Dissipation): Conditionally *Stable*
   (c) 3D Delta-Form Implicit Method is computationally efficient over equivalent unfactored scheme.
8. Work-horse algorithm for modern implicit codes: e.g. OVERFLOW